Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series
Differential Calculus #13

Logarithm Functions

In calculus we use base e so that we have:

$$(e^x)' = e^x$$

Suppose now that $y = e^x$. We can turn this around to calculate x from a given y.

When $y = e^x$ we write $x = \log_e y$ the **logarithm** of y to base e.

This is often written $x = \log y$ with the base e implied or simply as $x = \ln y$.

The convention is that if no base is mentioned you can assume it is e.

The logarithmic property

Where exponentials have the rule $e^{a+b} = e^a e^b$, for logarithms we get:

$$\log ab = \log a + \log b$$

Note: $\log(a + b)$ ***cannot*** be simplified.

This is why logarithms were first invented. You can multiple a and b by adding their logarithms. From 1620 up to about 1970, people used logarithms as an aid for multiplying numbers, taking powers and finding roots. Now we use electronic calculators and computers to do this arithmetic.

Why we still need logarithms

There are still many important reasons to understand logarithms. These centre on the fact that the logarithm is the inverse of the exponential.

Key: $\log m^t = t \log m$

Suppose you have invested \$1000 at 2.2% per annum. After t years you have:

$$A(t) = 1000 \times (1.022)^t$$

When will you have \$2,000?

Solve for t in:

$$2000 = 1000 \times (1.022)^t$$

$$(1.022)^t = 2$$

$$t \log(1.022) = \log 2$$

$$t = \frac{\log 2}{\log 1.022} = 31.9\ years$$

If you can find 6% this is:

$$\frac{\log 2}{\log 1.06} = 11.9\ years$$

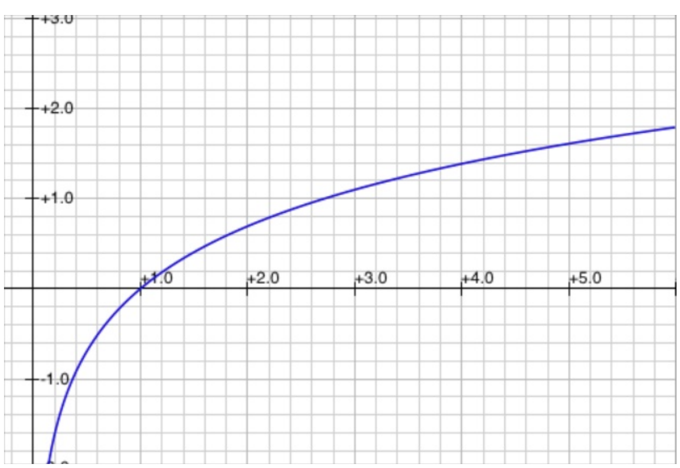
Exponential models are used all over the place.

- Finance
- Radioactive decay
- Heat flux
- Population growth
- Spread of disease
- Air pressure in the atmosphere

Logarithms are used in scientific measurements:

- Earthquake power (Richter)
- Hydrogen ion concentration (pH)
- Sound perception (Decibels Db)
- Star magnitudes

Shape of the Logarithm Curve



First, $\log x$ is only defined for $x > 0$.

$$\log 1 = 0$$
$$\lim_{x\to\infty} \log x = \infty$$
$$\lim_{x\to 0} \log x = -\infty$$

Derivative of $y = \log x$

If $y = \log x$ then $x = \exp y = e^y$.

Hence we have $e^{\log x} = x$.

Differentiate both sides of the equation.
We need the chain rule for the left side.

$$(e^{\log x})' = (x)'$$

$$e^{\log x}(\log x)' = 1$$

$$x(\log x)' = 1$$

$$(\log x)' = \frac{1}{x}$$

Summary:

1. When $y = e^x$ we write $x = \log_e y$
2. $x = e^{\log x} = \log e^x$ ➔ "log and exp are inverses"
3. $\log x = \log_e x = \ln x$
4. $\log ab = \log a + \log b$
5. $\log m^t = t \log m$
6. $(\log x)' = \frac{1}{x}$
7. We use logarithms whenever we use an exponential model.

Examples:

There is a general simplification that sometimes helps.

Since $x^3 = x \cdot x \cdot x$ we have

$$\log x^3 = \log x + \log x + \log x = 3 \log x$$

This is true for any power of x

$$\log x^m = m \log x$$

$$\log \frac{1}{x^4} = \log x^{-4} = -\,4 \log x$$

This is also true for something like x^x where the exponent is not constant.

Calculate the derivative of:

1. $y = \log(x + 3)$
2. $y = \log(x^3)$
3. $y = \log(x^2 + 1)$
4. $y = \log\,(\cos x)$
5. $y = \log(\sec x)$
6. $y = \tan(\log x)$
7. $y = x \log x - x$
8. $y = e^x \log x$

DON'T LOOK AT THE ANSWERS
UNTIL YOU HAVE MADE AN HONEST EFFORT TO SOLVE THE PROBLEMS.

All of these use: $\frac{d \log x}{dx} = \frac{1}{x}$

1. $y = \log(x+3)$

➔ $y' = \frac{1}{x+3}(x+3)' = \frac{1}{x+3}$

2. $y = \log(x^3)$

➔ $y = 3\log x$ So $y' = \frac{3}{x}$

3. $y = \log(x^2+1)$

➔ $y' = \frac{1}{x^2+1}(x^2+1)' = \frac{2x}{x^2+1}$

4. $y = \log(\cos x)$

➔ $y' = \frac{1}{\cos x}(\cos x)' = \frac{-\sin x}{\cos x} = -\tan x$

5. $y = \log(\sec x)$

➔ $y = \log\frac{1}{\cos x} = -\log\cos x$ so $y' = \tan x$

6. $y = \tan(\log x)$

➔ $y' = \sec^2(\log x)(\log x)' = \sec^2(\log x)\frac{1}{x}$

7. $y = x\log x - x$

➔ $y' = 1 \cdot \log x + x \cdot \left(\frac{1}{x}\right) - 1 = \log x + 1 - 1 = \log x$

8. $y = e^x \log x$

➔ $y' = e^x \log x + e^x(\frac{1}{x})$

Other bases:

We can write any positive base $r = e^{\log r}$

So $r^x = e^{(\log r)x}$

Similarly $\log_r x = \frac{\log x}{\log r}$

Then (using chain rule)

$$(r^x)' = \left(e^{x \log r}\right)' = e^{x \log r}(\log r) = (\log r) r^x$$

Exercise:

Suppose a flask of bacteria is growing so that the population doubles every hour. If there were originally 10^6 bacteria, write a formula for the number at time t in the future.

Solution: Measure time in hours. Since the doubling will mean multiplying by 2 as many times as there have been hours, in t hours we multiply by 2 in all t times. So we will multiply by 2^t. The formula for the population $P(t)$ at time t then looks like:

$$P(t) = m\, 2^t$$

We also know that $P(0) = m = 10^6$.
Here we use $2^0 = 1$.
The answer is $P(t) = 10^6\, 2^t$

If this question went on to ask for the rate of change of the population at 3 hours we would need the derivative:

$$P(t)' = \log 2\; 10^6 2^t$$

So $P'(3) = \log 2\; 10^6 2^3 = 5.545 \times 10^6$